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# The Klauder–Daubechies construction of the phase-space path integral and the harmonic oscillator

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## Abstract

The canonical operator quantization formulation corresponding to the Klauder–Daubechies construction of the phase-space path integral is considered. This formulation is explicitly applied and solved in the case of the harmonic oscillator, thereby illustrating in a manner complementary to Klauder and Daubechies’ original work some of the promising features offered by their construction of a quantum dynamics. The Klauder–Daubechies functional integral involves a regularization parameter eventually taken to vanish, which defines a new physical time scale. When extrapolated to the field theory context, besides providing a new regularization of short distance divergences, keeping a finite value for that time scale offers some tantalizing prospects when it comes to strong gravitational quantum systems.

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## 1. Introduction

The central role of the phase-space symplectic one-form in the quantization programme is well known and understood. No less important and crucial to the physical properties of the quantum system however is the role of an implicit phase-space Riemannian metric—albeit a ‘shadow phase-space metric’ [1–6] for what the classical world is concerned. This is convincingly

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argued by Klauder in an insightful and thought provoking paper [1] deserving to be much more widely known, which relies on prior work with Daubechies [7–12]. Making the role of this phase-space Riemannian metric explicit circumvents the ambiguities and difficulties inherent to the formal path integral definition of a quantized system. It even provides for a perfectly well-defined functional integral over continuous paths in phase space. One requirement that these two geometrical structures on phase space have to meet is that both the symplectic and Riemannian metrics define an identically normalized phase-space volume form.

The Klauder–Daubechies construction is achieved through stochastic calculus methods, involving a Wiener measure the diffusion parameter of which provides a regulator which, when eventually taken to infinity, reproduces the correct quantum mechanical amplitudes obeying the Schrödinger equation of the system. In terms closer to a physicist’s intuition perhaps, this Wiener measure is associated with the statistical Brownian motion of a particle propagating in the background Riemannian geometry of phase space, with a specific diffusion time scale taken eventually to vanish. One of the remarkable features of the Klauder–Daubechies construction of the phase-space path integral is its inherent manifest covariance under general canonical transformations in phase space, much in contradistinction to all other approaches leading to path integral representations of quantum amplitudes, or even to naive canonical operator quantization rules.

In order to make these statements somewhat more explicit, for the purpose of this introductory discussion only, let us assume a system of physical units such that all relevant parameters and scales are set to unity, inclusive of  $\hbar = 1$ , and let us restrict to a single degree of freedom system,  $q(t) \in \mathbb{R}$ , with canonically conjugate momentum,  $p(t) \in \mathbb{R}$ , obeying at the quantum level the Heisenberg algebra,  $[Q, P] = i\mathbb{1}$ ,  $Q^\dagger = Q$ ,  $P^\dagger = P$ . If  $|q, p\rangle$  denote the normalized canonical Weyl–Heisenberg quantum coherent states labelled by all classical phase-space states and associated with the normalized Fock vacuum  $|0\rangle$  such that  $(Q + iP)|0\rangle = 0$  [13], the Klauder–Daubechies path integral (KD-PI) representation of the associated matrix elements of the quantum system’s evolution operator of Hamiltonian  $\hat{H}_0(Q, P)$  is given in the form [1]

$$\begin{aligned} \langle q_f, p_f | e^{-iT\hat{H}_0} | q_i, p_i \rangle &= \lim_{\tau_0 \rightarrow 0^+} e^{C_0 \frac{T}{\tau_0}} \int_{(q(t_i), p(t_i))=(q_i, p_i)}^{(q(t_f), p(t_f))=(q_f, p_f)} \left[ \frac{\mathcal{D}q(t)\mathcal{D}p(t)}{2\pi} \right] \\ &\times e^{\int_{t_i}^{t_f} dt [\frac{1}{2}(\dot{q}p - q\dot{p}) - h(q, p)]} \times e^{-i\tau_0 \int_{t_i}^{t_f} dt \frac{1}{2}(\dot{q}^2 + \dot{p}^2)}, \end{aligned} \quad (1)$$

where the time interval  $T = t_f - t_i$  is such that  $T > 0$ , and  $\tau_0 > 0$  is a time scale regularization parameter. In this expression one integrates over all those paths in phase space possessing as end points those classical states associated with the external quantum coherent states. Furthermore,  $h(q, p)$  is the coherent state symbol representing the Hamiltonian operator through [13]

$$\hat{H}_0 = \int_{(\infty)} \frac{dq dp}{2\pi} |q, p\rangle h(q, p) \langle q, p|, \quad \int_{(\infty)} \frac{dq dp}{2\pi} |q, p\rangle \langle q, p| = \mathbb{1}. \quad (2)$$

To the lowest order in  $\hbar$ ,  $h(q, p)$  coincides with the classical Hamiltonian,  $H_0(q, p)$ . One thus recognizes in the phase of the first exponential factor inside the path integral the first-order Hamiltonian action of the system, inclusive of quantum corrections contributing in  $h(q, p)$ . In the absence of these corrections, that phase factor provides the usual naive formal definition of the quantized system through the phase-space path integral, a definition however, which is not free of ambiguities nor difficulties, among which a lack of covariance under phase-space canonical transformations. Note in particular the contribution to the Hamiltonian action in  $dt(\dot{q}p - q\dot{p})/2 = (dq p - q dp)/2 = K$ , which defines the symplectic one-form of the

phase-space symplectic geometry. The associated volume form,  $\omega = dK = dp \wedge dq$ , is thus normalized to unity.

However, the expression in (1) carries still two further  $\tau_0$ -dependent exponential factors. Returning to the very first one later on, the very last one inside the path integral is of a purely statistical character, being purely real Gaussian, in contradistinction to the previous pure phase factor of a purely quantum mechanical character. In effect the real Gaussian factor plays the role of a phase-space Wiener measure which regularizes, for any finite  $\tau_0 > 0$ , the ordinary naive path integral based on the purely imaginary (Gaussian and higher order) phase factor alone. Furthermore, one recognizes in the real Gaussian contribution precisely the Brownian motion of a particle in the background phase-space Euclidean geometry associated with the Weyl–Heisenberg algebra defined by the operators  $Q$  and  $P$  (other homogeneous geometries are also discussed in [1]). Note that the volume element associated with the Riemannian geometry with as metric a tensor given by the unit matrix is again normalized to unity, as is the volume form associated with the symplectic one-form involved in the pure phase factor.

Note that by having introduced the time scale  $\tau_0$ , the one-dimensional system with configuration space coordinate  $q$  and two-dimensional phase space  $(q, p)$  has been promoted to some two-dimensional system with configuration space  $(q, p)$ , hence a four-dimensional phase space, the dimensional reduction of which back to the space  $q$  is achieved only through the limit  $\tau_0 \rightarrow 0^+$ . As such the Lagrangian action for this effective two-dimensional system reads

$$\int_{t_i}^{t_f} dt \left[ \frac{1}{2} i\tau_0 (\dot{q}^2 + \dot{p}^2) + \frac{1}{2} (\dot{q}p - q\dot{p}) - h(q, p) \right]. \quad (3)$$

In this expression one recognizes the action of a particle of pure positive imaginary mass,  $m_0 = i\tau_0$ , moving in a two-dimensional Euclidean plane, subjected to a potential energy  $h(q, p)$ , as well as a velocity-dependent, hence magnetic, coupling defined by the symplectic one-form of the Hamiltonian formulation of the original system, as if the particle was coupled to a static homogeneous magnetic field perpendicular to the two-dimensional space  $(q, p)$ . Except for the mass factor which is not real, this is precisely a generalized Landau problem in phase space with interaction energy  $h(q, p)$  [2, 3]. As is well known, in the absence of this interaction energy, the energy levels of the quantized Landau problem are organized in infinitely degenerate discrete Landau levels, with a gap set by the ratio of the magnetic coupling to the mass. In the presence of the interaction energy  $h(q, p)$ , the Landau level degeneracies are lifted but states are still organized in discrete Landau sectors with a gap set by the same ratio. In the limit of a vanishing mass, namely in the present context the limit  $\tau_0 \rightarrow 0^+$ , this gap grows infinitely. Hence in order that not all Landau sectors decouple one has to adjust the quantum vacuum energy of the lowest Landau sector such that the energy of all states in that sector retains a finite energy in that limit. This is precisely the reason for the very first exponential factor in (1) multiplying the path integral,  $C_0$  being some normalization factor to be adjusted accordingly (which may be done up to an arbitrary finite contribution even when  $h(q, p) = 0$  [1]). And as a consequence, the surviving quantum states of the lowest Landau sector span the quantum Hilbert space of the original quantum system with the single degree of freedom  $q$ . Dimensional reduction in phase space is achieved for the quantum system by projecting onto its lowest Landau sector the extended quantized system, as defined by (1). Note that by the same token noncommutativity in the  $(q, p)$  space is induced once again through that projection, out of commuting operators  $(q, p)$  as configuration space coordinates for the extended dynamics. In essence, this is the genesis of the noncommutative Moyal plane of noncommutative quantum mechanics as well.

Incidentally, besides this intriguing possibility of having ‘extra dimensions’ introduced in a dynamics which are neither of a space- nor of a time-like character (as in Kaluza–Klein

or string theory contexts) but are rather of a phase-space character, the mixture of both purely quantum and statistical behaviours present in the formulation of a quantum dynamics as provided by the KD-PI construction in (1) reminds one of progress made by 't Hooft [14] with precisely such motivations in mind towards a deterministic formulation of quantum dynamics displaying at the same time a stochastic behaviour.

Until recently [15, 16] to the present authors' best knowledge, and in spite of all the potential interest offered by this approach to quantum dynamics, if only to illustrate explicitly the workings of (1) no actual evaluation of the KD-PI was available—certainly not for a finite value for  $\tau_0$ —even for as simple a test-bed system as the harmonic oscillator, the basis for all of perturbative relativistic quantum field theory. Certainly to the authors of the KD-PI it is clear—having proved it—that the quantum dynamics of the original system is recovered in the limit  $\tau_0 = 0$ . But if the formulation is to find practical applications, some explicit evaluations with finite  $\tau_0$  are most presumably useful. Furthermore, besides the path integral point of view on which the construction of (1) is based, a complementary understanding of the quantum properties of the extended system associated with the action in (3) from the canonical operator quantization point of view should prove to be of relevance as well, and could lead to further insight into the workings of the limit  $\tau_0 = 0$ . Finally, keeping the value for  $\tau_0$  finite may also be of interest in the context of deformations of algebraic structures associated with quantum dynamics in a more general setting, for instance that of quantum gravity and noncommutative geometries of spacetime [15–17].

The purpose of the present paper is not to justify the result in (1), but rather, by starting from it, to show explicitly that it indeed reproduces the correct quantum dynamics of the harmonic oscillator, and thereby to acquire greater familiarity with the meaning of the Klauder–Daubechies approach and the prospects it may offer. And since this has already been done in [15, 16] through a direct saddle-point evaluation of the path integral (1) for a finite  $\tau_0$  and in the limit  $\tau_0 = 0$ , the same issue is addressed here directly from the canonical operator quantization point of view, based on the  $\tau_0$  deformed effective action (3) of the system defined over the original phase space promoted to a two-dimensional configuration space. In the case of the one-dimensional harmonic oscillator, one is thus dealing with a Landau problem with pure positive imaginary mass subjected to a harmonic potential well. Even though the quantum solution for that system should be straightforward enough, its lack of unitarity and its properties under the limit  $\tau_0 = 0$  are sufficiently instructive to deserve a detailed analysis. At the same time, a broader and perhaps clearer understanding of the relevance and potential interest of the Klauder–Daubechies construction of the phase-space path integral is achieved.

The paper is organized as follows. In section 2, the canonical formulation associated with the extended action (3) is constructed. Section 3 then applies this formalism to the one-dimensional harmonic oscillator to construct the canonical quantization of its extended formulation, and its quantum solution, enabling thereby an explicit analysis of the limit  $\tau_0 = 0$  corresponding to the effective projection onto the lowest Landau sector of the system. Section 4 then addresses the evaluation of the projected quantum evolution operator for a finite value of  $\tau_0$  to compare with the saddle-point evaluation of [15, 16]. Finally, some conclusions are presented in section 5.

## 2. Canonical formulation of the extended system

First let us still consider an arbitrary one degree of freedom system, with canonically conjugate phase-space variables  $(q, p)$  and classical Hamiltonian  $H_0(q, p)$ , and reinstate dimensionful quantities, inclusive of all explicit factors of  $\hbar$ . In order to account for the different physical dimensions of the configuration space variable,  $q$ , and its conjugate momentum,  $p$ , let us also

introduce a constant factor  $\lambda_0$  having the dimension of mass times angular frequency. It proves then useful to work in terms of the following rescaled phase-space coordinates,  $\phi^a$  ( $a = 1, 2$ ), with

$$\phi^1 = \frac{1}{\sqrt{\lambda_0}} p, \quad \phi^2 = q \sqrt{\lambda_0}, \tag{4}$$

having the canonical Poisson brackets,  $\{\phi^a, \phi^b\} = -\epsilon^{ab}$ ,  $\epsilon^{ab}$  being the two-dimensional antisymmetric symbol with  $\epsilon^{12} = +1$ . Hence the classical Hamiltonian first-order action of the system reads

$$S_0[\phi^a] = \int dt \left[ \frac{1}{2} (\dot{q}p - q\dot{p}) - H_0(q, p) \right] = \int dt \left[ \frac{1}{2} \epsilon_{ab} \phi^a \dot{\phi}^b - H_0(\phi^a) \right]. \tag{5}$$

For what the extended system is concerned, the function  $H_0(\phi^a)$  gets replaced by the symbol  $h(\phi^a) = h(q, p)$ , while the associated Lagrangian action reads

$$S[\phi^a] = \int dt \left[ \frac{1}{2} i\tau_0 \delta_{ab} \dot{\phi}^a \dot{\phi}^b + \frac{1}{2} \epsilon_{ab} \phi^a \dot{\phi}^b - h(\phi^a) + E_0 \right], \tag{6}$$

$\delta_{ab}$  being the phase-space Euclidean metric, and  $E_0$  some ( $\hbar$ -dependent) constant to be adjusted later on in order to retain quantum states of finite energy in the limit  $\tau_0 \rightarrow 0^+$  in the manner explained previously.

Developing a classical canonical formulation corresponding to this Lagrangian action as such is problematic. Indeed, even when initial or boundary conditions for  $\phi^a$  are specified to be real valued, because of the pure imaginary mass term trajectories solving the associated classical Euler–Lagrange equations of motions are bound to become complex valued, hence also the momentum variables,  $p_a$ , conjugate to the configuration space ones,  $\phi^a$ . At the quantum level it would therefore appear to be unjustified to associate with both these quantities operators that are self-adjoint.

However, one should keep in mind that the above Lagrangian action for the extended system only contributes inside a quantum path integral of the form

$$\int [\mathcal{D}\phi^a(t)] e^{\frac{i}{\hbar} S[\phi^a]}, \tag{7}$$

where integration is taken over real paths in the real-valued configuration space,  $\phi^a(t)$ , not involving therefore the complex-valued classical trajectories (unless one considers an evaluation of the integral through contour deformations into the complex plane, as is done effectively in a saddle-point evaluation [15, 16]). In terms of this path integral it becomes possible to introduce real-valued variables,  $p_a$ , canonically conjugate to the real configuration space ones,  $\phi^a$ , as auxiliary variables for some well-defined real Gaussian integrals, thereby bringing the path integral into the canonical first-order form, namely,

$$\int [\mathcal{D}\phi^a(t)] e^{\frac{i}{\hbar} S[\phi^a]} = \int [\mathcal{D}\phi^a(t) \mathcal{D}p_a(t)] e^{\frac{i}{\hbar} \int dt [\dot{\phi}^a p_a - H(\phi^a, p_a)]}, \tag{8}$$

where

$$H(\phi^a, p_a) = \frac{1}{2i\tau_0} \delta^{ab} \left( p_a + \frac{1}{2} \epsilon_{ac} \phi^c \right) \left( p_b + \frac{1}{2} \epsilon_{bd} \phi^d \right) + h(\phi^a) - E_0 \tag{9}$$

(the absolute normalization of the functional integration measures is left unspecified). In particular, note that the Gaussian integrals over  $p_a$  are real and well defined precisely because the mass parameter,  $m_0 = i\tau_0$ , is pure positive imaginary.

Clearly it is this latter form of the path integral which defines the canonical formulation of the extended system, with real canonically conjugate phase space variables ( $\phi^a, p_a$ ) and

canonical Hamiltonian function  $H(\phi^a, p_a)$ . As is well known, such a path integral is associated with an operator realization over some Hilbert space providing a representation of the following extended Heisenberg algebra:

$$[\hat{\phi}^a, \hat{p}_b] = i\hbar\delta_b^a\mathbb{I}, \quad \hat{\phi}^{a\dagger} = \hat{\phi}^a, \quad \hat{p}_a^\dagger = \hat{p}_a, \quad (10)$$

with indeed Hermitian operators, and note well, also commuting  $\hat{\phi}^a$ , namely  $\hat{q}$  and  $\hat{p}$  operators. Hence rather than considering the path integral in (1), an equivalent realization of the same extended quantum system for a finite  $\tau_0$  value is defined by this operator algebra and the quantum Hamiltonian

$$\hat{H} = \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_a + \frac{1}{2}\epsilon_{ac}\hat{\phi}^c\right)\left(\hat{p}_b + \frac{1}{2}\epsilon_{bd}\hat{\phi}^d\right) + h(\hat{\phi}^a) - E_0. \quad (11)$$

It is thus the eigenspectrum of this operator that needs to be understood as a function of  $\tau_0$ , as well as its behaviour in the limit  $\tau_0 = 0$ . Note, however, that because of the pure imaginary mass parameter, this operator is not Hermitian,  $\hat{H}^\dagger \neq \hat{H}$ ; hence, the quantum dynamics of the extended quantum system is not unitary, for any finite  $\tau_0 > 0$ . In particular, its eigenspectrum proves to be complex but with a dependence on  $\tau_0$  such that for those states that survive the limit  $\tau_0 = 0$ , their limiting energy eigenvalues are real once again and coincide with the eigenspectrum of the original unitary quantized system. This very point may thus be studied explicitly for a function  $h(\phi^a)$  which, for example, is purely quadratic (and linear) in the variables  $\phi^a$ , namely essentially the case of the one-dimensional harmonic oscillator.

Let us henceforth consider a harmonic oscillator of mass  $m$  and angular frequency  $\omega_0 > 0$ , with the choice  $\lambda_0 = m\omega_0$ . The Hamiltonian then reads

$$H_0(q, p) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega_0^2q^2 = \frac{1}{2}\omega_0\delta_{ab}\phi^a\phi^b. \quad (12)$$

Except for an additive constant proportional to  $\hbar$  which may be absorbed in the choice for  $E_0$ , in this case the symbol  $h(q, p)$  for the quantum Hamiltonian  $\hat{H}_0$  coincides with  $H_0(q, p)$ . Consequently, the operator quantization of the extended system in the case of the harmonic oscillator is defined by the Heisenberg algebra in (10) as well as the following quantum Hamiltonian:

$$\hat{H} = \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_a + \frac{1}{2}\epsilon_{ac}\hat{\phi}^c\right)\left(\hat{p}_b + \frac{1}{2}\epsilon_{bd}\hat{\phi}^d\right) + \frac{1}{2}\omega_0\delta_{ab}\hat{\phi}^a\hat{\phi}^b - E_0, \quad (13)$$

the diagonalization of which we now address.

### 3. The ordinary harmonic oscillator

#### 3.1. A bi-module of Fock-like algebras

Given the Hermitian operators  $\hat{\phi}^a$  and  $\hat{p}_a$ , let us introduce first the following Fock operators,

$$a_a = \frac{1}{2\sqrt{\hbar}}(\hat{\phi}^a + 2i\hat{p}_a), \quad a_a^\dagger = \frac{1}{2\sqrt{\hbar}}(\hat{\phi}^a - 2i\hat{p}_a), \quad (14)$$

which define the tensor product of two Fock algebras,

$$[a_a, a_b^\dagger] = \delta_{ab}\mathbb{I}. \quad (15)$$

Next, consider the following helicity Fock operators:

$$a_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2), \quad a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger \pm ia_2^\dagger), \quad (16)$$

such that

$$[a_\pm, a_\pm^\dagger] = \mathbb{I}, \quad [a_\pm, a_\mp^\dagger] = 0. \quad (17)$$

The inverse relations are

$$\begin{aligned}\hat{\phi}^1 &= \sqrt{\frac{\hbar}{2}}(a_+ + a_- + a_+^\dagger + a_-^\dagger), & \hat{p}_1 &= -\frac{i}{2}\sqrt{\frac{\hbar}{2}}(a_+ + a_- - a_+^\dagger - a_-^\dagger), \\ \hat{\phi}^2 &= i\sqrt{\frac{\hbar}{2}}(a_+ - a_- - a_+^\dagger + a_-^\dagger), & \hat{p}_2 &= \frac{1}{2}\sqrt{\frac{\hbar}{2}}(a_+ - a_- + a_+^\dagger - a_-^\dagger),\end{aligned}\quad (18)$$

with in particular,

$$\hat{p}_1 + \frac{1}{2}\hat{\phi}^2 = -i\sqrt{\frac{\hbar}{2}}(a_- - a_-^\dagger), \quad \hat{p}_2 - \frac{1}{2}\hat{\phi}^1 = -\sqrt{\frac{\hbar}{2}}(a_- + a_-^\dagger). \quad (19)$$

To construct an abstract representation of these algebraic structures, consider now a normalized Fock vacuum,  $|\Omega\rangle$ , for the helicity Fock operators,

$$a_\pm|\Omega\rangle = 0, \quad \langle\Omega|\Omega\rangle = 1, \quad (20)$$

with the following orthonormalized states spanning the Hilbert space of the extended quantum system,

$$|n_+, n_-; \Omega\rangle = \frac{1}{\sqrt{n_+!n_-!}}(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}|\Omega\rangle, \quad \langle n_+, n_-; \Omega|m_+, m_-; \Omega\rangle = \delta_{n_+, m_+}\delta_{n_-, m_-}, \quad (21)$$

where  $n_+, n_- = 0, 1, 2, \dots$ , hence with the resolution of the unit operator,

$$\sum_{n_+, n_- = 0}^{\infty} |n_+, n_-; \Omega\rangle\langle n_+, n_-; \Omega| = \mathbb{I}. \quad (22)$$

Incidentally, these are the states that diagonalize the quantum operator  $\hat{H}$  in the absence of the interaction coupling  $h(\hat{\phi}^a)$ , leading to Landau levels labelled by  $n_- = 0, 1, \dots$  and infinitely degenerate in  $n_+ = 0, 1, \dots$ . In particular, the lowest Landau level  $|n_+, n_- = 0; \Omega\rangle$  ( $n_+ = 0, 1, \dots$ ) will turn out to define the subspace of the Hilbert space of the extended quantum system which coincides with the Hilbert space of the original quantum system, namely the lowest Landau sector in the presence of the interaction energy  $h(\hat{\phi}^a)$  in the limit when  $\tau_0 \rightarrow 0^+$ . For that reason it is useful to already introduce the projector onto that subspace of quantum states of the extended system,

$$\mathbb{P}_0 = \sum_{n_+ = 0}^{\infty} |n_+, n_- = 0; \Omega\rangle\langle n_+, n_- = 0; \Omega|, \quad \mathbb{P}_0^2 = \mathbb{P}_0, \quad \mathbb{P}_0^\dagger = \mathbb{P}_0. \quad (23)$$

Note that we then have for the projected operators generating the Heisenberg algebra in the extended Hilbert space,

$$\mathbb{P}_0(\hat{p}_1 + \frac{1}{2}\hat{\phi}^2)\mathbb{P}_0 = 0, \quad \mathbb{P}_0(\hat{p}_2 - \frac{1}{2}\hat{\phi}^1)\mathbb{P}_0 = 0, \quad (24)$$

showing that after projection only the projected coordinates  $\mathbb{P}_0\hat{\phi}^a\mathbb{P}_0 = \bar{\phi}^a$  are independent operators, with commutation relations,

$$[\bar{\phi}^a, \bar{\phi}^b] = -i\hbar\epsilon^{ab}\mathbb{P}_0. \quad (25)$$

Hence, indeed on the projected subspace one recovers the Heisenberg algebra of the original quantum system, even though within the extended Hilbert space the phase-space position operators  $\hat{\phi}^a$  commute with each other.

However, the above states do not diagonalize the total Hamiltonian  $\hat{H}$  in the presence of the interaction  $h(\hat{\phi}^a)$ , even for the harmonic oscillator. In the latter case, other linear

combinations of the basic operators  $\hat{\phi}^a$  and  $\hat{p}_a$  are required. For that purpose, let us introduce two specific real quantities,  $R_0$  and a phase  $\varphi_0$ , defined by the following relation:

$$R_0^2 e^{2i\varphi_0} \equiv 1 + 4i\omega_0\tau_0, \quad R_0 > 0, \quad 0 \leq \varphi_0 < \frac{\pi}{4}, \quad (26)$$

as well as the complex variable  $\rho$  and its complex conjugate  $\bar{\rho}$ ,

$$\rho = \sqrt{R_0} e^{\frac{1}{2}i\varphi_0}, \quad \bar{\rho} = \sqrt{R_0} e^{-\frac{1}{2}i\varphi_0}. \quad (27)$$

Note that in the limit  $\tau_0 \rightarrow 0^+$ , or in the absence of the coupling  $\omega_0$ ,  $R_0$  and  $\rho$  both go to unity while  $\varphi_0$  vanishes.

Consider then the operators

$$A_a = \frac{1}{2\sqrt{\hbar}} \left( \rho \hat{\phi}^a + \frac{2i}{\rho} \hat{p}_a \right), \quad B_a = \frac{1}{2\sqrt{\hbar}} \left( \rho \hat{\phi}^a - \frac{2i}{\rho} \hat{p}_a \right), \quad (28)$$

as well as their adjoints,

$$A_a^\dagger = \frac{1}{2\sqrt{\hbar}} \left( \bar{\rho} \hat{\phi}^a - \frac{2i}{\bar{\rho}} \hat{p}_a \right), \quad B_a^\dagger = \frac{1}{2\sqrt{\hbar}} \left( \bar{\rho} \hat{\phi}^a + \frac{2i}{\bar{\rho}} \hat{p}_a \right), \quad (29)$$

which are such that

$$[A_a, B_b] = \delta_{ab} \mathbb{I} = [B_a^\dagger, A_b^\dagger]. \quad (30)$$

Had it not been for the fact that  $\rho$  is a complex quantity, the operators  $A_a$  and  $B_a$  would have been adjoints of one another. We have, for instance,

$$A_a^\dagger = \cos \varphi_0 B_a - i \sin \varphi_0 A_a, \quad B_a^\dagger = \cos \varphi_0 A_a - i \sin \varphi_0 B_a. \quad (31)$$

Furthermore the operators  $A_a$  and  $B_a$  almost coincide with  $a_a$  and  $a_a^\dagger$  above, respectively, but only if  $\rho = 1$ , namely whenever  $\omega_0\tau_0 = 0$ . Clearly, the operators  $A_a, B_a$  and their adjoints may be expressed as linear combinations of  $a_a$  and  $a_a^\dagger$ .

Finally, let us introduce the helicity combinations

$$\begin{aligned} A_\pm &= \frac{1}{\sqrt{2}}(A_1 \mp iA_2), & B_\pm &= \frac{1}{\sqrt{2}}(B_1 \pm iB_2), \\ A_\pm^\dagger &= \frac{1}{\sqrt{2}}(A_1^\dagger \pm iA_2^\dagger), & B_\pm^\dagger &= \frac{1}{\sqrt{2}}(B_1^\dagger \mp iB_2^\dagger), \end{aligned} \quad (32)$$

such that

$$[A_\pm, B_\pm] = \mathbb{I} = [B_\pm^\dagger, A_\pm^\dagger], \quad (33)$$

as well as

$$\begin{aligned} A_\pm^\dagger &= \cos \varphi_0 B_\pm - i \sin \varphi_0 A_\mp, & A_\pm &= \cos \varphi_0 B_\pm^\dagger + i \sin \varphi_0 A_\mp^\dagger, \\ B_\pm^\dagger &= \cos \varphi_0 A_\pm - i \sin \varphi_0 B_\mp, & B_\pm &= \cos \varphi_0 A_\pm^\dagger + i \sin \varphi_0 B_\mp^\dagger. \end{aligned} \quad (34)$$

Expressing these operators in terms of  $a_\pm$  and  $a_\pm^\dagger$ , one finds

$$\begin{aligned} A_\pm &= \frac{\rho + \rho^{-1}}{2} a_\pm + \frac{\rho - \rho^{-1}}{2} a_\mp^\dagger, & A_\pm^\dagger &= \frac{\bar{\rho} + \bar{\rho}^{-1}}{2} a_\pm^\dagger + \frac{\bar{\rho} - \bar{\rho}^{-1}}{2} a_\mp, \\ B_\pm &= \frac{\rho + \rho^{-1}}{2} a_\pm^\dagger + \frac{\rho - \rho^{-1}}{2} a_\mp, & B_\pm^\dagger &= \frac{\bar{\rho} + \bar{\rho}^{-1}}{2} a_\pm + \frac{\bar{\rho} - \bar{\rho}^{-1}}{2} a_\mp^\dagger. \end{aligned} \quad (35)$$

The Fock-like algebraic relations in (33) are very much similar to those of ordinary Fock algebras, except for the fact that the operators  $B_\pm$  and  $A_\pm$  (on the one hand, or their adjoints on the other hand) are not adjoints of one another. Yet, a representation theory may be constructed in very much the same way, leading to dual states we shall refer to as *A*- and *B*-Fock states. This representation is built on *A*- and *B*-Fock vacua,  $|\Omega_A\rangle$  and  $|\Omega_B\rangle$ , respectively, such that

$$A_\pm |\Omega_A\rangle = 0, \quad B_\pm^\dagger |\Omega_B\rangle = 0. \quad (36)$$

By an appropriate choice of phases and normalizations, it is always possible to assume that the inner product of these two states is set to unity,

$$\langle \Omega_A | \Omega_B \rangle = 1 = \langle \Omega_B | \Omega_A \rangle. \quad (37)$$

The  $A$ -Fock states are then defined by

$$|N_+, N_-; \Omega_A\rangle = \frac{1}{\sqrt{N_+! N_-!}} B_+^{N_+} B_-^{N_-} |\Omega_A\rangle, \quad (38)$$

while for the  $B$ -Fock states,

$$|N_+, N_-; \Omega_B\rangle = \frac{1}{\sqrt{N_+! N_-!}} (A_+^\dagger)^{N_+} (A_-^\dagger)^{N_-} |\Omega_B\rangle, \quad (39)$$

where  $N_+, N_- = 0, 1, 2, \dots$ . As a matter of fact, since the operators  $A_\pm, B_\pm$  and their adjoints are linear combinations of the Fock operators  $a_\pm$  and  $a_\pm^\dagger$ , it is clear that either set of states,  $|N_+, N_-; \Omega_A\rangle$  or  $|N_+, N_-; \Omega_B\rangle$ , spans the entire Hilbert space of the quantum extended system. More specifically, each of these two sets provides a basis of that space, these two bases being in fact dual to one another,

$$\langle N_+, N_-; \Omega_A | M_+, M_-; \Omega_B \rangle = \delta_{N_+, M_+} \delta_{N_-, M_-} = \langle N_+, N_-; \Omega_B | M_+, M_-; \Omega_A \rangle. \quad (40)$$

Consequently, one also has the following resolutions of the unit operator:

$$\sum_{N_+, N_- = 0}^{\infty} |N_+, N_-; \Omega_A\rangle \langle N_+, N_-; \Omega_B| = \mathbb{I} = \sum_{N_+, N_- = 0}^{\infty} |N_+, N_-; \Omega_B\rangle \langle N_+, N_-; \Omega_A|. \quad (41)$$

In other words, the three sets of states,  $|n_+, n_-; \Omega\rangle, |N_+, N_-; \Omega_A\rangle$  and  $|N_+, N_-; \Omega_B\rangle$ , define three different bases of the same extended Hilbert space, with the basis  $|n_+, n_-; \Omega\rangle$  being self-dual since orthonormalized, while the other two bases are dual to one another.

Note that the action of the  $A_\pm$  and  $B_\pm$  operators on the  $A$ -Fock states, on the one hand, and of the  $B_\pm^\dagger$  and  $A_\pm^\dagger$  operators on the  $B$ -Fock states, on the other hand, is precisely like that of ordinary annihilation and creation Fock operators, respectively, on ordinary Fock states. In particular, the  $A$ -Fock states  $|N_+, N_-; \Omega_A\rangle$  (resp.,  $B$ -Fock states  $|N_+, N_-; \Omega_B\rangle$ ) are eigenstates of the operators  $B_\pm A_\pm$  (resp.,  $A_\pm^\dagger B_\pm^\dagger$ ) with eigenvalues  $N_\pm$ .

Given the identities (35) relating the different Fock-like operators, it should be clear that the relations between these three different bases are obtained as Bogoliubov transformations. Introducing the complex parameter

$$\lambda = \frac{\rho - \rho^{-1}}{\rho + \rho^{-1}}, \quad \bar{\lambda} = \frac{\bar{\rho} - \bar{\rho}^{-1}}{\bar{\rho} + \bar{\rho}^{-1}}, \quad (42)$$

a little analysis shows that the  $A$ - and  $B$ -Fock vacua are given as

$$|\Omega_A\rangle = \left( \frac{2}{\rho + \rho^{-1}} \right) e^{-\lambda a_+^\dagger a_-^\dagger} |\Omega\rangle, \quad |\Omega_B\rangle = \left( \frac{2}{\bar{\rho} + \bar{\rho}^{-1}} \right) e^{-\bar{\lambda} a_+^\dagger a_-^\dagger} |\Omega\rangle, \quad (43)$$

and similarly

$$|\Omega_B\rangle = N_B(\varphi_0) e^{i \tan \varphi_0 B_+ B_-} |\Omega_A\rangle, \quad |\Omega_A\rangle = N_A(\varphi_0) e^{-i \tan \varphi_0 A_+^\dagger A_-^\dagger} |\Omega_B\rangle, \quad (44)$$

$N_A(\varphi_0)$  and  $N_B(\varphi_0)$  being two normalization factors whose evaluation is not required here,

$$N_A^{-1}(\varphi_0) = \langle \Omega_B | e^{-i \tan \varphi_0 A_+^\dagger A_-^\dagger} |\Omega_B\rangle, \quad N_B^{-1}(\varphi_0) = \langle \Omega_A | e^{i \tan \varphi_0 B_+ B_-} |\Omega_A\rangle. \quad (45)$$

These different representations relating the different Fock vacua as coherent helicity pairing excitations of one another thus establish that indeed all three sets of Fock states provide complete bases of the same extended Hilbert space in which to diagonalize the total quantum Hamiltonian  $\hat{H}$ .

Finally, note that in the limit where  $\tau_0 \rightarrow 0^+$ , all the three sets of Fock states then coalesce into a single set, namely the states  $|n_+, n_-; \Omega\rangle$  ( $n_+, n_- = 0, 1, \dots$ ), since then all three Fock vacua become identical to  $|\Omega\rangle$  while we have the following correspondences for the creation and annihilation operators:

$$A_{\pm} \rightarrow a_{\pm}, \quad B_{\pm} \rightarrow a_{\pm}^{\dagger}, \quad A_{\pm}^{\dagger} \rightarrow a_{\pm}^{\dagger}, \quad B_{\pm}^{\dagger} \rightarrow a_{\pm}. \quad (46)$$

### 3.2. The energy spectrum

With the previous representation theory of the extended Hilbert space at hand, diagonalization of the total Hamiltonian (13) of the extended system is readily achieved. In terms of the operators introduced above, a little substitution easily finds

$$\hat{H} = \hbar \frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0} B_- A_- + \hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \left( B_+ A_+ + \frac{1}{2} \right) + \left( \hbar \frac{R_0 e^{i\varphi_0} + 1}{4i\tau_0} - E_0 \right). \quad (47)$$

Obviously, the  $A$ -Fock states,  $|N_+, N_-; \Omega_A\rangle$ , are the eigenstates of that operator, while those of its adjoint,  $\hat{H}^{\dagger} \neq \hat{H}$ , are the  $B$ -Fock states,  $|N_+, N_-; \Omega_B\rangle$ . Furthermore the subtraction constant  $E_0$  needs to be adjusted as follows:

$$E_0 = \hbar \frac{R_0 e^{i\varphi_0} + 1}{4i\tau_0} - \Delta E_0(\omega_0, \tau_0), \quad \lim_{\tau_0 \rightarrow 0^+} \Delta E_0(\omega_0, \tau_0) = 0, \quad (48)$$

where the function  $\Delta E_0(\omega_0, \tau_0)$  is *a priori* otherwise arbitrary (it may even be complex for a finite value of  $\tau_0$ ), and in fact is of the form

$$\Delta E_0(\omega_0, \tau_0) = \hbar \omega_0 \Delta \mathcal{E}_0(\omega_0 \tau_0), \quad (49)$$

$\Delta \mathcal{E}_0(\omega_0 \tau_0)$  being a function of the product  $(\omega_0 \tau_0)$  only which vanishes when that argument vanishes. In the limit  $\tau_0 \rightarrow 0^+$ , clearly then only the lowest Landau sector with  $N_- = 0$  retains finite energy values, namely the states  $|N_+, N_- = 0; \Omega_A\rangle \rightarrow |n_+ = N_+, n_- = 0; \Omega\rangle$  for  $\hat{H}$  and  $|N_+, N_- = 0; \Omega_B\rangle \rightarrow |n_+ = N_+, n_- = 0; \Omega\rangle$  for  $\hat{H}^{\dagger}$ , with  $N_+ = 0, 1, \dots$ .

Given that choice for the subtraction constant  $E_0$ , the complex energy spectrum of the system, for a finite value of  $\tau_0 > 0$ , is given as

$$\begin{aligned} \hat{H} |N_+, N_-; \Omega_A\rangle &= E(N_+, N_-) |N_+, N_-; \Omega_A\rangle, \\ \hat{H}^{\dagger} |N_+, N_-; \Omega_B\rangle &= \bar{E}(N_+, N_-) |N_+, N_-; \Omega_B\rangle, \end{aligned} \quad (50)$$

with

$$E(N_+, N_-) = \hbar \frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0} N_- + \hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \left( N_+ + \frac{1}{2} \right) + \Delta E_0(\omega_0, \tau_0), \quad (51)$$

while  $\bar{E}(N_+, N_-)$  stands for the complex conjugate of  $E(N_+, N_-)$ . In particular, the lowest Landau sector energy eigenvalues are

$$E(N_+, N_- = 0) = \hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \left( N_+ + \frac{1}{2} \right) + \Delta E_0(\omega_0, \tau_0). \quad (52)$$

### 3.3. The $\tau_0 \rightarrow 0^+$ limit

In the absence of the interaction energy  $\hbar(\hat{\phi}^a)$ , namely when  $\omega_0 = 0$ , the energy spectrum reduces to

$$\omega_0 = 0 : \quad E(N_+, N_-) = \frac{\hbar}{i\tau_0} N_-, \quad (53)$$

displaying the infinite degeneracy in  $N_+ = 0, 1, 2, \dots$  of the Landau levels labelled by  $N_- = 0, 1, 2, \dots$  and separated by a gap  $\hbar/(i\tau_0)$  as expected, then corresponding to the states  $|n_+ = N_+, n_- = N_-; \Omega\rangle$ . In the limit  $\tau_0 = 0$ , only the lowest Landau level retains a finite (vanishing) energy.

When the interaction energy  $h(\hat{\phi}^a)$  is included, the gap between Landau sectors is determined by the quantity

$$\hbar \frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0}, \tag{54}$$

which in the limit  $\tau_0 \rightarrow 0^+$  behaves as

$$\hbar \frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0} \underset{\tau_0 \rightarrow 0^+}{\simeq} \frac{\hbar}{i\tau_0} + \hbar\omega_0 + \dots \tag{55}$$

Hence once again it is the scale  $\hbar/(i\tau_0)$  which sets the leading contribution to that gap, which diverges in the limit  $\tau_0 \rightarrow 0^+$ . Consequently, only the Landau sector with  $N_- = 0$  retains a finite energy in that limit. Furthermore within a given Landau sector, the spacing between states is determined by the second relevant quantity

$$\hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0}, \tag{56}$$

which in the limit  $\tau_0 \rightarrow 0^+$  behaves as

$$\hbar \frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \underset{\tau_0 \rightarrow 0^+}{\simeq} \hbar\omega_0 + \dots \tag{57}$$

Hence, in that limit, the energy spectrum behaves as

$$E(N_+, N_-) \underset{\tau_0 \rightarrow 0^+}{\simeq} \frac{\hbar}{i\tau_0} (1 + i\omega_0\tau_0 + \dots) N_- + (\hbar\omega_0 + \dots) \left(N_+ + \frac{1}{2}\right) + \dots \tag{58}$$

Those states retaining a finite energy in that limit belong only to the lowest Landau sector with  $N_- = 0$ ,

$$\lim_{\tau_0 \rightarrow 0^+} E(N_+, N_- = 0) = \hbar\omega_0 \left(N_+ + \frac{1}{2}\right). \tag{59}$$

In this expression one recognizes the real energy spectrum of the harmonic oscillator, including its quantum vacuum energy, the corresponding energy eigenstates being the Fock states  $|n_+ = N_+, n_- = 0; \Omega\rangle$ . Hence indeed the subspace of the extended Hilbert space of the extended system spanned by the lowest Landau sector in the limit  $\tau_0 = 0$  determines the Hilbert space of the original quantum system, in the present case that of the harmonic oscillator.

To show that the remaining Landau sectors do decouple from the energy spectrum in the limit  $\tau_0 = 0$ , it suffices to consider the quantum evolution operator of the extended system. Given the spectral resolution of the unit operator in terms of the eigenstates of the Hamiltonian operator,  $\hat{H}$ , and its adjoint, one has for the evolution operator with  $T = t_f - t_i > 0$ ,

$$e^{-\frac{i}{\hbar}T\hat{H}} = \sum_{N_+, N_- = 0}^{\infty} |N_+, N_-; \Omega_A\rangle e^{-\frac{i}{\hbar}TE(N_+, N_-)} \langle N_+, N_-; \Omega_B|. \tag{60}$$

Using the above expansion in  $\tau_0$  for  $E(N_+, N_-)$ , one thus finds that all the states with  $N_- \geq 1$  decouple exponentially in the considered limit,

$$\lim_{\tau_0 \rightarrow 0^+} e^{-\frac{i}{\hbar}T\hat{H}} \underset{T \geq 0}{=} \sum_{N_+ = 0}^{\infty} |N_+, N_- = 0; \Omega\rangle e^{-i\omega_0 T(N_+ + 1/2)} \langle N_+, N_- = 0; \Omega|. \tag{61}$$

Hence indeed all but the states belonging to the lowest Landau sector have decoupled from the dynamics of the extended system in the limit  $\tau_0 \rightarrow 0^+$ , leaving over precisely the Hilbert space of the ordinary harmonic oscillator with the correct energy spectrum and quantum time evolution operator. The states  $|n_+, n_- = 0; \Omega\rangle$  correspond exactly to the usual Fock states  $|n_+\rangle$  of the harmonic oscillator with energy spectrum  $E(n_+) = \hbar\omega_0(n_+ + 1/2)$ ,  $|n_+, n_- = 0; \Omega\rangle \equiv |n_+\rangle$ .

This conclusion is thus in full accord with the general discussion and results of [1] within the functional integral setting, but achieved in the specific case of the harmonic oscillator and using rather operator quantization techniques. As a matter of fact, a similar analysis based on the operator quantization of the extended system for whatever initial system and given (polynomial [1]) Hamiltonian  $H_0(q, p)$  is possible, leading of course to the same general conclusion [18, 19].

The above analysis has thus also established that the limit  $\tau_0 \rightarrow 0^+$  enforces the projection effected by the operator  $\mathbb{P}_0$  introduced previously, thereby leading back to noncommuting Hermitian projected phase-space operators,  $Q = \mathbb{P}_0\hat{q}\mathbb{P}_0$  and  $P = \mathbb{P}_0\hat{p}\mathbb{P}_0$ , obeying the usual Heisenberg algebra as it should,  $[Q, P] = i\hbar\mathbb{P}_0$ , of which the projected Hilbert space spanned by the Fock states  $|n\rangle \equiv |n, 0; \Omega\rangle$  provides the usual Fock space representation of that Heisenberg algebra. However, before the projection is effected, as two-dimensional configuration space operators, the unprojected coordinates  $\hat{q}$  and  $\hat{p}$  acting on the extended Hilbert space are commuting operators. This feature, unique to the Klauder–Daubechies construction of the phase-space path integral which is covariant under canonical transformations of the original system, can be put to use to exploit at the quantum level all the advantages of classical action-angle transformations for systems which are integrable in the Liouville sense and which possess nonperturbative configurations [18, 19].

## 4. The deformed harmonic oscillator

### 4.1. A deformed quantum dynamics

The previous discussion has thus established that one has for the  $\mathbb{P}_0$  projected evolution operator of the extended system, when  $T > 0$ ,

$$\lim_{\tau_0 \rightarrow 0^+} \mathbb{P}_0 e^{-\frac{i}{\hbar}T\hat{H}}\mathbb{P}_0 = \lim_{\tau_0 \rightarrow 0^+} e^{-\frac{i}{\hbar}T\hat{H}}, \quad (62)$$

the latter quantity then reproducing the quantum evolution operator of the original system. However, since  $\mathbb{P}_0$  effects the projection onto the Hilbert space of the original quantum system, the projected evolution operator may be worth considering also for a finite value of  $\tau_0$ ,

$$T > 0 : \quad \mathbb{U}(T) = \mathbb{P}_0 e^{-\frac{i}{\hbar}T\hat{H}}\mathbb{P}_0 \neq \lim_{\tau_0 \rightarrow 0^+} \mathbb{P}_0 e^{-\frac{i}{\hbar}T\hat{H}}\mathbb{P}_0, \quad (63)$$

knowing that in the limit  $\tau_0 = 0$  this operator reproduces the correct evolution operator of the original quantum system,

$$U(T) = \lim_{\tau_0 \rightarrow 0^+} \mathbb{U}(T) = \lim_{\tau_0 \rightarrow 0^+} e^{-\frac{i}{\hbar}T\hat{H}}. \quad (64)$$

Keeping  $\tau_0$  finite for  $\mathbb{U}(T)$  thus induces a deformed quantum dynamics inside the Hilbert space of the original quantum system as compared to the operator  $U(T)$ . Such a deformation may be of physical interest, in a spirit comparable to that which suggests to consider noncommutative deformations of the geometrical properties of spacetime in attempts towards formulations for a quantum theory of gravity through deformations of quantum algebras [15–17]. Nevertheless,

it should be pointed out that for a finite value of  $\tau_0$ , because of the irreversible character of its Brownian motion component, such a dynamics is no longer unitary,

$$\mathbb{U}^\dagger(T) \neq \mathbb{U}^{-1}(T), \quad \mathbb{U}^\dagger(T)\mathbb{U}(T) \neq \mathbb{P}_0, \quad \mathbb{U}(T)\mathbb{U}^\dagger(T) \neq \mathbb{P}_0, \quad (65)$$

and thus cannot preserve quantum probabilities, or more correctly in the present context, the total occupation number (the sum of the occupation densities over all quantum states of the system). Nor does it meet the usual convolution property under consecutive time evolution intervals,

$$\mathbb{U}(T_2) \cdot \mathbb{U}(T_1) \neq \mathbb{U}(T_2 + T_1), \quad \mathbb{P}_0 e^{-\frac{i}{\hbar}T_2\hat{H}}\mathbb{P}_0 \cdot \mathbb{P}_0 e^{-\frac{i}{\hbar}T_1\hat{H}}\mathbb{P}_0 \neq \mathbb{P}_0 e^{-\frac{i}{\hbar}(T_2+T_1)\hat{H}}\mathbb{P}_0. \quad (66)$$

Hence such a proposal raises a series of interpretational issues, which we shall not attempt to address here. However, let us point out that when extrapolated to a quantum field theory context [20], a finite  $\tau_0$  value provides in effect a regularization of short-distance singularities, akin to a soft exponential cut-off in the momentum of quantum states, indeed so efficient that all quantum amplitudes for whatever field theory in a perturbative expansion, even including general relativity, are ultra-violet finite (the only potential source of trouble being some tadpole contributions, which may always be dealt with by a proper choice of quantum Hamiltonian). The combination of the time scale  $\tau_0$ —expected to be extremely small in the physical world if non-vanishing—and of the Planck time in a quantum gravitational context,  $\tau_{\text{Planck}} = \sqrt{\hbar G_N/c^5} \simeq 10^{-43}$  s—irrespective of whether these two time scales should prove to be unrelated or not—may thus offer some tantalizing prospects for strongly gravitationally interacting quantum systems [15], a physical situation in which perhaps the requirements of unitarity and Lorentz invariance may be relaxed to some slight degree for what concerns experimentally unexplored extreme regimes. Whatever the case may be, at least a non-vanishing time scale  $\tau_0$  provides yet another regularization of short-distance quantum dynamics for local field theories whose usefulness is worth exploring.

As a matter of fact the projected operator  $\mathbb{U}(T)$  has already been computed [15, 16] directly from the KD-PI in (1) using a saddle-point approach for what is indeed a purely Gaussian functional integral in the case of the harmonic oscillator. Here rather, we shall exploit the operator solution constructed above to reproduce the same result, making it readily explicit that the deformed quantum dynamics remains diagonal in the Fock state basis of the harmonic oscillator.

#### 4.2. The projected evolution operator

Since the operator of interest is of the form

$$\mathbb{U}(T) \stackrel{T>0}{=} \sum_{n_+, m_+=0}^{\infty} |n_+, 0; \Omega\rangle \langle n_+, 0; \Omega| e^{-\frac{i}{\hbar}T\hat{H}} |m_+, 0; \Omega\rangle \langle m_+, 0; \Omega|, \quad (67)$$

while the eigenstates of  $\hat{H}$  (resp.,  $\hat{H}^\dagger$ ) are  $|N_+, N_-; \Omega_A\rangle$  (resp.,  $|N_+, N_-; \Omega_B\rangle$ ), one first needs to consider the following change of basis matrix elements:

$$\langle n_+, 0; \Omega|N_+, N_-; \Omega_A\rangle, \quad \langle m_+, 0; \Omega|N_+, N_-; \Omega_B\rangle. \quad (68)$$

Using the definition of the  $A$ -Fock states  $|N_+, N_-; \Omega_A\rangle$  and the representation of  $|\Omega_A\rangle$  as a coherent helicity pairing excitation of  $|\Omega\rangle$ , a detailed evaluation of the first matrix element finds the following result:

$$\langle n_+, 0; \Omega|N_+, N_-; \Omega_A\rangle = \left(\frac{2}{\rho + \rho^{-1}}\right) \left(\frac{2}{\rho + \rho^{-1}}\right)^{N_+} \left(\frac{\rho - \rho^{-1}}{2}\right)^{N_-} \sqrt{\frac{N_+!}{n_+!N_-!}} \delta_{N_+, N_-+n_+}. \quad (69)$$

In a similar fashion,

$$\langle m_+, 0; \Omega | N_+, N_-; \Omega_B \rangle = \left( \frac{2}{\bar{\rho} + \bar{\rho}^{-1}} \right) \left( \frac{2}{\bar{\rho} + \bar{\rho}^{-1}} \right)^{N_+} \left( \frac{\bar{\rho} - \bar{\rho}^{-1}}{2} \right)^{N_-} \sqrt{\frac{N_+!}{m_+! N_-!}} \delta_{N_+, N_- + m_+}. \quad (70)$$

It then readily follows that the matrix elements of the deformed evolution operator  $\mathbb{U}(T)$  in the Fock state basis of the harmonic oscillator are diagonal,

$$\langle n | \mathbb{U}(T) | \ell \rangle \equiv \langle n, 0; \Omega | \mathbb{U}(T) | \ell, 0; \Omega \rangle = \delta_{n,\ell} \langle n | \mathbb{U}(T) | n \rangle. \quad (71)$$

Using the above results, a direct evaluation of the diagonal matrix element then leads to

$$\langle n | \mathbb{U}(T) | n \rangle = e^{-\frac{i}{\hbar} T \Delta E_0} e^{-i(n+\frac{1}{2})\alpha_+} F^{n+1}(T), \quad (72)$$

where

$$\alpha_+ = \omega_0 T \frac{R_0 e^{i\varphi_0} - 1}{2i\omega_0 \tau_0}, \quad \alpha_- = \omega_0 T \frac{R_0 e^{i\varphi_0} + 1}{2i\omega_0 \tau_0}, \quad (73)$$

and

$$\frac{1}{F(T)} = \left( \frac{\rho + \rho^{-1}}{2} \right)^2 - \left( \frac{\rho - \rho^{-1}}{2} \right)^2 e^{-i(\alpha_+ + \alpha_-)}. \quad (74)$$

In order to bring this matrix element to a more amenable form, in terms of the two quantities  $R_0$  and  $\varphi_0$  defined previously already through the identification (26) let us introduce the following further notations:

$$R = \sqrt{\frac{1}{2} (R_0^2 + 1)}, \quad S = \frac{1}{2} (R + 1), \quad (75)$$

which are such that

$$R - 1 = 2 \frac{\omega_0^2 \tau_0^2}{R^2 S}, \quad \frac{\omega_0 \tau_0}{RS} = \sqrt{1 - \frac{1}{S}}, \quad \cos \varphi_0 = \frac{R}{R_0}, \quad \sin \varphi_0 = \frac{2\omega_0 \tau_0}{R_0 R}. \quad (76)$$

It then follows that

$$i\alpha_+ = T \frac{R - 1}{2\tau_0} + i \frac{\omega_0 T}{R}, \quad i(\alpha_+ + \alpha_-) = T \frac{R}{\tau_0} + 2i \frac{\omega_0 T}{R}, \quad (77)$$

as well as

$$\rho^2 = R + 2i \frac{\omega_0 \tau_0}{R}, \quad \rho^{-2} = \frac{R^2 - 2i\omega_0 \tau_0}{R_0^2 R}, \quad (78)$$

and finally,

$$\frac{1}{F(T)} = e^{-\frac{R}{\tau_0} T} e^{-2i \frac{\omega_0}{R} T} + S \frac{R + 2i\omega_0 \tau_0}{R^2 + 2i\omega_0 \tau_0} (1 - e^{-\frac{R}{\tau_0} T} e^{-2i \frac{\omega_0}{R} T}). \quad (79)$$

Hence we have so far

$$\langle n | \mathbb{U}(T) | n \rangle = e^{-\frac{i}{\hbar} T \Delta E_0(\omega_0, \tau_0)} e^{-i \frac{\omega_0 T}{R} (n+\frac{1}{2})} e^{-\frac{R-1}{2\tau_0} T (n+\frac{1}{2})} F^{n+1}(T). \quad (80)$$

Since

$$\lim_{T \rightarrow +\infty} F(T) = \frac{1}{S} \frac{R^2 + 2i\omega_0 \tau_0}{R + 2i\omega_0 \tau_0}, \quad (81)$$

in order that the asymptotic time limit  $T \rightarrow +\infty$  leaves over at least one of the matrix elements  $\langle n | \mathbb{U}(T) | n \rangle$  with a finite and non-vanishing occupation, given that  $R > 1$  this can only be the

case for the Fock vacuum  $|n = 0\rangle = |\Omega\rangle$ , which requires then to specify the choice for the arbitrary function  $\Delta E_0(\omega_0, \tau_0)$  as follows:

$$\Delta E_0(\omega_0, \tau_0) = -i\hbar \frac{R-1}{4\tau_0} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} -\frac{1}{2} i\hbar \omega_0^2 \tau_0 + \dots, \tag{82}$$

indeed a pure imaginary quantity but such that it vanishes in the limit  $\tau_0 = 0$ , as it should. Correspondingly, we have for the energy subtraction constant  $E_0$ ,

$$E_0 = \hbar \frac{R_0 e^{i\omega_0} - R + 2}{4i\tau_0} = \hbar \frac{1}{2i\tau_0} + \hbar \frac{\omega_0}{2R}. \tag{83}$$

Incidentally, the exact same choice had to be made in [15, 16] for precisely the same reason. Note also that in the absence of the interaction energy  $\hbar(\hat{\phi}^a)$ , namely when  $\omega_0 = 0$ , the value  $E_0 = \hbar/(2i\tau_0)$  coincides precisely with the one specified in [1] for the factor  $e^{C_0 T/\tau_0}$  in (1).

In conclusion, the final expression for the relevant matrix elements, which agrees with the result obtained through a functional integral calculation [15, 16], is

$$\langle n | \mathbb{U}(T) | \ell \rangle = \delta_{n,\ell} \cdot e^{-i\frac{\omega_0}{R} T(n+\frac{1}{2})} e^{-n\frac{R-1}{2\tau_0} T} F^{n+1}(T) \equiv \delta_{n,\ell} \cdot \mathbb{U}_n(T); \tag{84}$$

hence,

$$\mathbb{U}(T) = \sum_{n=0}^{\infty} |n\rangle e^{-i\frac{\omega_0}{R} T(n+\frac{1}{2})} e^{-n\frac{R-1}{2\tau_0} T} F^{n+1}(T) \langle n| = \sum_{n=0}^{\infty} |n\rangle \mathbb{U}_n(T) \langle n|. \tag{85}$$

Note that we have

$$\lim_{T \rightarrow 0^+} \mathbb{U}(T) = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{P}_0, \quad \lim_{\tau_0 \rightarrow 0^+} \mathbb{U}(T) = \sum_{n=0}^{\infty} |n\rangle e^{-i\omega_0 T(n+\frac{1}{2})} \langle n| = U(T), \tag{86}$$

as it should, while

$$\lim_{T \rightarrow +\infty} \mathbb{U}(T) = e^{-\frac{1}{2} i\frac{\omega_0}{R} T} \frac{1}{S} \frac{R^2 + 2i\omega_0\tau_0}{R + 2i\omega_0\tau_0} |\Omega\rangle \langle \Omega|, \tag{87}$$

thus displaying how because of the Brownian motion contribution to the quantum dynamics when  $\tau_0 \neq 0$ , whatever the initial state of the system it eventually decays to the Fock vacuum with a specific factor rescaling the initial occupation of that particular state.

Before commenting on the significance of these results, let us consider how the original Heisenberg algebra of phase-space operators is deformed in the time evolved picture of the system, because of a non-vanishing value for  $\tau_0 > 0$ . Defining quantum operators,  $A(t_f)$ , in the Heisenberg picture in the usual way but in terms of the projected evolution operator,  $\mathbb{U}(T)$  with  $T = t_f - t_i > 0$ , as

$$A(t_f) = \mathbb{U}^\dagger(T) A(t_i) \mathbb{U}(T), \tag{88}$$

a direct calculation in the case of the (projected) position and momentum operators,  $Q(t_i) = \mathbb{P}_0 \hat{q}(t_i) \mathbb{P}_0$  and  $P(t_i) = \mathbb{P}_0 \hat{p}(t_i) \mathbb{P}_0$  with  $[Q(t_i), P(t_i)] = i\hbar \mathbb{P}_0$ , finds indeed a deformed Heisenberg algebra

$$\begin{aligned} [Q(t_f), P(t_f)] &= i\hbar |0\rangle F_0^3 D^3(T) e^{-\frac{R-1}{\tau_0} T} \langle 0| \\ &+ i\hbar \sum_{n=1}^{\infty} |n\rangle F_0^{2n+1} D^{2n+1}(T) e^{-(2n-1)\frac{R-1}{\tau_0} T} [(n+1)F_0^2 D^2(T) e^{-2\frac{R-1}{\tau_0} T} - n] \langle n|. \end{aligned} \tag{89}$$

In this expression the quantities  $F_0$  and  $D(T)$  are defined according to the relation

$$|F(T)|^2 = F_0 \cdot D(T), \tag{90}$$

where

$$F_0 = \frac{1}{S^2} \frac{R^4 + 4\omega_0^2 \tau_0^2}{R^2 + 4\omega_0^2 \tau_0^2} \quad (91)$$

and

$$\frac{1}{D(T)} = 1 + 2 \left( \frac{R-1}{R+1} \right) e^{-\frac{R}{\tau_0} T} \cos 2 \left( \frac{\omega_0}{R} T + \varphi_0 \right) + \left( \frac{R-1}{R+1} \right)^2 e^{-\frac{2R}{\tau_0} T}. \quad (92)$$

Note that we have

$$\lim_{T \rightarrow 0^+} [Q(t_f), P(t_f)] = i\hbar \mathbb{P}_0, \quad \lim_{\tau_0 \rightarrow 0^+} [Q(t_f), P(t_f)] = i\hbar \mathbb{P}_0, \quad (93)$$

as it should, while

$$\lim_{T \rightarrow +\infty} [Q(t_f), P(t_f)] = 0. \quad (94)$$

The reason why in the asymptotic time limit,  $T \rightarrow +\infty$ , the two phase-space operators  $Q(t_f)$  and  $P(t_f)$  end up commuting with one another as in the classical system is that in that limit all quantum states of the harmonic oscillator except for its Fock vacuum have exponentially decayed to zero, as shown explicitly by (87).

### 4.3. Physical implications

More precisely, given an initial quantum state

$$|\psi, t_i\rangle = \sum_{n=0}^{\infty} |n\rangle \psi_n(t_i), \quad \psi_n(t_i) \in \mathbb{C}, \quad \sum_{n=0}^{\infty} |\psi_n(t_i)|^2 < \infty, \quad (95)$$

its configuration at time  $t_f$  with  $T = t_f - t_i > 0$  is

$$\begin{aligned} |\psi, t_f\rangle &= \mathbb{U}(T) |\psi, t_i\rangle = \sum_{n=0}^{\infty} |n\rangle \mathbb{U}_n(T) \psi_n(t_i) = \sum_{n=0}^{\infty} |n\rangle \psi_n(t_f), \\ \psi_n(t_f) &= \mathbb{U}_n(t_f - t_i) \psi_n(t_i). \end{aligned} \quad (96)$$

Consequently, the time evolution of the occupation densities of the Fock eigenstates of the harmonic oscillator is determined by

$$|\psi_n(t_f)|^2 = |\mathbb{U}_n(t_f - t_i)|^2 \cdot |\psi_n(t_i)|^2. \quad (97)$$

Based on the expressions above, one has

$$|\mathbb{U}_n(T)|^2 = |F(T)|^{2(n+1)} e^{-n \frac{R-1}{\tau_0} T}, \quad (98)$$

namely

$$|\mathbb{U}_n(T)|^2 = F_0^{n+1} D^{n+1}(T) e^{-n \frac{R-1}{\tau_0} T}, \quad (99)$$

where the quantities  $F_0$  and  $D(T)$  are given in (91) and (92), respectively.

Stochastic Brownian motion leads to so efficient a statistical decoherence of the quantum system that whatever dynamics there is to begin with, it totally decays away. All that remains is a rescaled occupation of the initial ground state occupation of the system. Given the asymptotic values

$$\lim_{T \rightarrow +\infty} |\mathbb{U}_{n=0}(T)|^2 = \frac{1}{S^2} \frac{R^4 + 4\omega_0^2 \tau_0^2}{R^2 + 4\omega_0^2 \tau_0^2}, \quad \lim_{T \rightarrow +\infty} |\mathbb{U}_{n \geq 1}(T)|^2 = 0, \quad (100)$$

the time asymptotics of the Fock state occupations is such that

$$\lim_{t_f \rightarrow +\infty} |\psi_{n=0}(t_f)|^2 = \frac{1}{S^2} \frac{R^4 + 4\omega_0^2 \tau_0^2}{R^2 + 4\omega_0^2 \tau_0^2} |\psi_{n=0}(t_i)|^2, \quad \lim_{t_f \rightarrow +\infty} |\psi_{n \geq 1}(t_f)|^2 = 0. \quad (101)$$

In its large time behaviour, the dynamics of the (non-interacting closed) system which is irreversible provided  $\tau_0$  is non-vanishing however small its value is such that the Hilbert space of the quantum system thus becomes effectively one dimensional, being aligned along the direction of the oscillator Fock vacuum  $|\Omega\rangle = |0\rangle$  only. All other excited Fock states  $|n\rangle$  decouple by decay (without being coupled to some external environment or interaction) with a hierarchy of lifetimes determined by  $\tau_+^{(n)} = \tau_0/(n(R-1))$ ,  $n = 1, 2, \dots$

More specifically, first one observes an oscillatory pattern contributing both to the overall phase factor proportional to  $(n+1/2)$  in  $|\mathbb{U}_n(T)\rangle$  and to the function  $F(T)$ , and thus to its modulus squared  $|F(T)|^2 = F_0 D(T)$  in  $|\mathbb{U}_n(T)\rangle^2$ . The periodicity of this pattern is set by a rescaling of the proper time scale of the oscillator by the factor  $R$ , namely by the following effective angular frequency:

$$\omega_{\text{effective}} = \frac{\omega_0}{R} < \omega_0. \quad (102)$$

Besides this oscillatory pattern, the time dependence of the Fock state occupations, modulated by  $|\mathbb{U}_n(T)|^2$ , is furthermore governed by two more real exponential time scales, the first of which modulates the factor  $|F(T)|^2$  and the second modulates the exponential in time normalization of  $|\mathbb{U}_n(T)\rangle^2$  for  $n \geq 1$ ,

$$\tau_- = \frac{\tau_0}{R} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} \tau_0 + \dots, \quad \tau_+^{(n)} = \frac{1}{n} \frac{\tau_0}{R-1} = \frac{1}{n} \frac{R^2 S}{2\omega_0^2 \tau_0} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} \frac{1}{n} \frac{1}{2\omega_0^2 \tau_0} + \dots \quad (103)$$

or, when measured in units either of the characteristic time scale of the oscillator,  $1/\omega_0$ , or the intrinsic time scale  $\tau_0$  of the quantum deformation of its dynamics,

$$\omega_0 \tau_- = \frac{\omega_0 \tau_0}{R} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} \omega_0 \tau_0 + \dots, \quad \omega_0 \tau_+^{(n)} = \frac{1}{n} \frac{\omega_0 \tau_0}{R-1} = \frac{1}{n} \frac{R^2 S}{2\omega_0 \tau_0} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} \frac{1}{n} \frac{1}{2\omega_0 \tau_0} + \dots, \\ \frac{\tau_-}{\tau_0} = \frac{1}{R} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} 1 + \dots, \quad \frac{\tau_+^{(n)}}{\tau_0} = \frac{1}{n} \frac{1}{R-1} = \frac{1}{n} \frac{R^2 S}{2(\omega_0 \tau_0)^2} \stackrel{\tau_0 \rightarrow 0^+}{\simeq} \frac{1}{n} \frac{1}{2(\omega_0 \tau_0)^2} + \dots \quad (104)$$

For a given value of  $\omega_0 \tau_0$ , and provided  $n$  is small enough such that  $\tau_- < \tau_+^{(n)}$  (which is always the case for  $n = 1$  at least), for any given Fock state  $|n\rangle$  there are then effectively three time windows characteristic of different regimes for the deformed quantum dynamics, namely  $0 \leq T \leq \tau_-$ ,  $\tau_- \leq T \leq \tau_+^{(n)}$  and  $\tau_+^{(n)} \leq T < \infty$  [15, 16]. To describe these windows it is relevant to consider the value of the characteristic time scale of the system,  $1/\omega_0$ , relative to the time scale of the deformation, namely the quantity  $1/(\omega_0 \tau_0)$  (note that if the physical system under consideration does not carry any characteristic time scale, for instance a free particle, no deviation from ordinary unitary quantum dynamics is present even when  $\tau_0 \neq 0$ ). Since ordinary quantum behaviour is recovered in the limit  $\tau_0 \rightarrow 0^+$ , when the quantity  $1/(\omega_0 \tau_0)$  is extremely large, for all practical purposes the quantum behaviour of the system does not significantly differ from that of ordinary quantum mechanics, at least up to the time scale  $\tau_+^{(n)}$  for each of those Fock states  $|n\rangle$  such that  $\tau_+^{(n)} > \tau_-$ . In the time window  $\tau_- \leq T \leq \tau_+^{(n)}$ , only a very small time-dependent rescaling of the Fock state occupation occurs; the larger the value of  $1/(\omega_0 \tau_0)$ , the smaller the rescaling. Since if indeed non-vanishing in the physical world the actual value of  $\tau_0$  is expected to be on the order of the Planck time, some  $10^{-43}$  s, while in comparison experimental conditions have not yet observed extremely high-intensity excitations of modes of large enough frequencies for particle and interaction fields, it seems fair to assume that until now all experiments conducted in laboratories have remained inside this

‘ordinary quantum physics window’ (this does not include violent astrophysical phenomena in strong gravitational quantum regimes that may be observed). It is only by moving into time scales  $1/\omega_0$  becoming comparable to  $\tau_0$ , that the time window for ordinary quantum mechanics begins to grow narrow enough that the deformed quantum dynamics of the system may start displaying deviations from ordinary unitary quantum behaviour, and thereby enable at least experimental upper bounds to be set on the deformation parameter  $\tau_0$ .

When reaching such a regime, which is then essentially also the situation for those Fock states  $|n\rangle$  with  $n$  sufficiently large such that now  $\tau_+^{(n)} < \tau_-$ , as well as for the time window  $\tau_+^{(n)} \leq T < \infty$  even in the discussion above, the telltale signs for the lack of a unitary quantum dynamics are, first, the total decoherence of the dynamics decaying ultimately to its ground state (on a time scale which is the smaller the larger is  $\omega_0\tau_0$ ), and second, the time-dependent rescaling or renormalization of the occupation density of that ground state and of the excited states at intermediate times, with in particular for the ground state an asymptotic in time rescaling of its occupation is given by the quantity

$$F_0 = \frac{1}{S^2} \frac{R^4 + 4\omega_0^2\tau_0^2}{R^2 + 4\omega_0^2\tau_0^2}. \quad (105)$$

The behaviour of the latter factor as a function of  $1/(\omega_0\tau_0)$  is noteworthy [15, 16],

$$F_0 \xrightarrow{1/(\omega_0\tau_0) \rightarrow 0} 4 \frac{1}{\omega_0\tau_0} + \dots, \quad F_0 \xrightarrow{1/(\omega_0\tau_0) \rightarrow +\infty} 1 + \frac{2}{(1/(\omega_0\tau_0))^2} + \dots. \quad (106)$$

Hence, as  $\tau_0 \rightarrow 0^+$ , the population rescaling factor  $F_0$  keeps on approaching the unit value which it has when  $\tau_0 = 0$ , *but from above*, which means that as  $1/(\omega_0\tau_0)$  decreases  $F_0$  keeps on growing ever larger than unity, until it reaches a maximal value lying above unity ( $F_0^{\max} \simeq 1.079$  for  $1/(\omega_0\tau_0) \simeq 2.591$ ) and from which further on, as  $1/(\omega_0\tau_0)$  still keeps decreasing,  $F_0$  starts decreasing as well, then passes the unit value, to finally reach a vanishing value in the limit that  $1/(\omega_0\tau_0)$  also vanishes. Consequently given a value for  $\tau_0$ , for an angular frequency larger than a certain threshold,  $\omega_{\text{threshold}}(\tau_0)$ , the survival occupation density of even the Fock vacuum is always less than its initial value, while for  $\omega_0$  values less than  $\omega_{\text{threshold}}(\tau_0)$ , the survival occupation density is always larger than its initial value. Nonetheless in all circumstances all excited Fock states end up not being populated at all at asymptotic times. Within a quantum field theory context, especially for the gravitational field, such behaviour clearly implies some tantalizing prospects for dynamics at the smallest spacetime scales, leading to an effective coarse-graining of spacetime geometry since this geometry may only be probed through interacting quantum fields.

## 5. Conclusions

This paper considered the canonical operator quantization formalism corresponding to the functional integral of the Klauder–Daubechies construction of the phase-space path integral [1]. The latter formulation introduces a regularization parameter, equivalent to a new time scale  $\tau_0 > 0$ , such that in the limit where it vanishes the construction reproduces the correct quantum dynamics of the system. This result was demonstrated explicitly from the operator representation of the same construction, in the specific case of the harmonic oscillator, thereby highlighting from a different and complementary point of view the inner workings of the Klauder–Daubechies approach to quantum dynamics.

In effect, this approach promotes the original system to the dynamics of an extended one of which the configuration space is the phase space of the original system, equipped not only with that phase-space’s symplectic geometry but also a Riemannian metric with the identical

volume form. The latter structure is related to a Brownian motion component added to the quantum dynamics of the original system, such that when the Brownian motion regularization is taken away again, only the original quantum system survives. This formulation offers a number of advantages, not least of which is its manifest covariance under general canonical transformations of the phase-space parametrization, which may be put to efficient use to develop new nonperturbative quantization techniques [18, 19]. Furthermore the extended regularizing dynamics is of the form of a generalized Landau problem in phase space, with a pure positive imaginary mass set by the time scale parameter  $\tau_0$ . In this respect, the Klauder–Daubechies construction comes in close resonance with present day developments in noncommutative geometry and quantum mechanics, most of which are inspired precisely by the Landau problem in the plane in the massless limit [17].

The operator formulation of the Klauder–Daubechies construction should also make it possible to extend it to systems with more than a single degree of freedom, one first case of interest being precisely the Landau problem itself and its associated noncommutative geometry of the Moyal plane. But beyond that, relativistic quantum field theories with their short-distance divergences in perturbation theory are another case in point. Indeed, the operator technique is well adapted to keep the value of  $\tau_0$  finite throughout, which is possibly a choice of physical relevance in the spirit of deformations of quantum algebraic structures, which however then reveals some appealing as well as some not so appealing new features. If only for that purpose, a finite  $\tau_0$  provides a new type of short-distance regularization in local quantum field theory taming all short-distance divergences. On the other hand, unitarity and Lorentz invariance are then lost at time scales less than  $\tau_0$ , with however a suppression of dynamics precisely on those scales as well which are bound to induce an effective coarse-graining of spacetime geometry in strong gravitational quantum systems. In the latter context, the status of initial cosmological singularities, or the issue of trans-Planckian energies in black hole radiation are open issues that come to mind, which could be addressed within the Klauder–Daubechies framework for quantum dynamics.

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